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Superspace geometry: the exact uncertainty relationship between complementary aspects

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Abstract. After reviewing the standard uncertainty relations due to Heisenberg, Robertson, and Schrödinger, as well as the relations of Deutsch, and of Maassen and Uffink—including the so-called entropic relations—we present a complete account of the uncertainty relationship between complementary aspects in terms of superspace geometry, an approach not hitherto employed. Two incompatible properties $A=\Sigma_\alpha A_\alpha|\alpha\rangle\langle\alpha|$ and $N=\Sigma_n N_n|n\rangle\langle n|$ belong to a pair of complementary aspects defined by two orthonormal bases $\{|\alpha\rangle\}$ and $\{|n\rangle\}$ in the Hilbert space $\mathscr{H}$. If the state is $|\psi\rangle$, then $P(\alpha)=|\langle\alpha|\psi\rangle|^2$ is the probability of obtainining the value A_α in a measurement of A, and $P(n)=|\langle n|\psi\rangle|^2$ is the probability of obtaining the value N_n in a measurement of N. The two aspects are characterised, relative to $|\psi\rangle$, by the numbers (so-called purities): $\pi_\varphi=\Sigma_\alpha P(\alpha)^2$ and $\pi_\ell=\Sigma_n P(n)^2$, both ≤ 1. We give a complete characterisation of the uncertainty relationship between A and N (more precisely: between their aspects) in terms of the range of joint values of (π_φ, π_ℓ) for arbitrary initial states (pure as well as mixed). A theorem of Lenard is given an alternative proof, employing only elementary (superspace) geometry. The results depend on two angles, $\phi_{\mathrm{m}}=$ minimal angle, and $\phi_{\mathrm{M}}=$ maximal angle between the two aspects. Exact expressions for ϕ_{m} and ϕ_{M} are obtained in terms of the overlap matrix $\Lambda=\{\Lambda_{\alpha n}\}=\{|\langle\alpha|n\rangle|^2\}$. As a corollary we find the uncertainty relation for a pure state $|\psi\rangle$

$$\pi_\varphi+\pi_\ell\leq\left(1+\frac{1}{g}\right)+\left(1-\frac{1}{g}\right)\cos\phi_{\mathrm{m}}$$

(where $g=\dim\mathscr{H}$), and a sharper one for mixed states. $\pi_\varphi+\pi_\ell=2$ is obtainable if and only if the intersection of the aspects holds a pure state. If $\phi_{\mathrm{m}}=\pi/2$ (maximal incompatibility), then $\pi_\varphi+\pi_\ell\leq 1+1/g$ is a special case of a stronger relation: $\Sigma_{\mu=0}^{g}\pi^{(\mu)}=2$, which one obtains for $g+1$ maximally incompatible aspects by means of a theorem of Ivanović.

1. Introduction

To measure a pair of incompatible observables can be regarded as investigating two complementary aspects of a system. As is well known, the system must be prepared twice, in the same state, and each measurement must be performed on a separate replica. The outcomes agree (statistically) with two alternative predictions, made by quantum theory, on the basis of the original state.

Let two such predictions be the expected statistical dispersions, Δq and Δp, for a pair of canonically conjugate properties, q and p. Then Δq and Δp are statements about the nature of the original state, as viewed from the alternative perspectives of the q- and p-aspects. The Heisenberg (1927) uncertainty relation

$$\Delta q\Delta p\geq\tfrac{1}{2}\hbar \tag{1}$$

can be said to characterise these aspects as such, insofar as it applies to arbitrary states.

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Considerable progress has recently been made with respect to such a direct characterisation of aspects in general. We first recall some important results. Then we present the exact uncertainty relationship which is the subject of the present work.

Let us explain this aim in more detail. Suppose X and Y are predictions characterising alternative aspects, e.g. $X = \Delta q$ and $Y = \Delta p$. An uncertainty relation is an inequality $f(X, Y) \geqslant 0$, where equality defines a curve in the X-Y plane: $f(X, Y) = 0$. In general this bound does not coincide with the boundary of the permissible region of (X, Y) over the set of all states (or over a subset of states). Equation (1) represents a special case in which bound and boundary may coincide (e.g. minimal-uncertainty states of the harmonic oscillator). But generally an *a priori* commitment to a particular f prevents this. Hence, by aiming at the exact uncertainty relationship we shall understand that the boundary is to be identified, for specific X and Y (to be defined for our present project in section 1.3). Within this scope, defined by the chosen X and Y, the uncertainty relationship provides a complete characterisation of the relationship between different aspects.

1.1. The Robertson-Schrödinger inequality

Until recently, the best available uncertainty relations for systems with finite-dimensional Hilbert spaces were those of Robertson (1929) and Schrödinger (1930). Let $\mathbb{W}$ be the state of a system, a positive definite, self-adjoint operator with trace $\mathrm{Tr}(\mathbb{W}) = 1$, and let the mean value functional be $\langle \cdot \rangle \equiv \mathrm{Tr}(\mathbb{W} \cdot)$. Then for two Hermitian operators, $\mathbb{A}$ and $\mathbb{B}$, the variances $(\Delta A)^2 \equiv \langle \mathbb{A}^2 \rangle - \langle \mathbb{A} \rangle^2$ and $(\Delta B)^2 \equiv \langle \mathbb{B}^2 \rangle - \langle \mathbb{B} \rangle^2$ are bounded according to

$$\Delta A \Delta B \geqslant |\langle \mathbb{A}\mathbb{B} \rangle - \langle \mathbb{A} \rangle \langle \mathbb{B} \rangle| \geqslant \tfrac{1}{2} |\langle [\mathbb{A}, \mathbb{B}] \rangle|. \tag{2}$$

The first inequality is due to Schrödinger, and the second to Robertson. Although the historically subsequent uncertainty relation of Schrödinger is sharper, it does not appear to be generally available. The proof is short enough that we should pause to display it.

Since $\langle \mathbb{A}\mathbb{B} \rangle = \mathrm{Tr}(\sqrt{\mathbb{W}}\mathbb{A}\mathbb{B}\sqrt{\mathbb{W}})$ is an inner product for $\mathbb{A}\sqrt{\mathbb{W}}$ and $\mathbb{B}\sqrt{\mathbb{W}}$ as Hilbert-Schmidt operators (cf, for example, Dunford and Schwartz 1963), or a semi-inner product for bounded $\mathbb{A}$ and $\mathbb{B}$ (cf, for example, Conway 1985), the Cauchy-Bunyakowski-Schwarz inequality gives $\langle \mathbb{A}^2 \rangle \langle \mathbb{B}^2 \rangle \geqslant |\langle \mathbb{A}\mathbb{B} \rangle|^2$. Replacing $\mathbb{A}$ by $\mathbb{A} - \langle \mathbb{A} \rangle \mathbb{1}$, etc, gives the Schrödinger inequality. For Hermitian operators

$$|\langle \mathbb{A}\mathbb{B} \rangle|^2 = \tfrac{1}{4} |\langle \{\mathbb{A}, \mathbb{B}\} \rangle + \langle [\mathbb{A}, \mathbb{B}] \rangle|^2 = \tfrac{1}{4} |\langle \{\mathbb{A}, \mathbb{B}\} \rangle|^2 + \tfrac{1}{4} |\langle [\mathbb{A}, \mathbb{B}] \rangle|^2.$$

Deleting the anticommutator yields the Robertson inequality. So the Schrödinger inequality is sharper.

Angular momentum is a typical example. Let $\vec{P} \equiv \langle \vec{\sigma} \rangle$ for the Pauli operators $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$. Then $\Delta\sigma_x \Delta\sigma_y \geqq |\mathrm{i} P_z - P_x P_y| = \{P_z^2 + P_x^2 P_y^2\}^{1/2} \geqslant |P_z|$.

Let the system's Hilbert space $\mathbb{H}$ have dimension $g = \dim(\mathbb{H})$. If $g = \infty$ the operators may be canonically conjugate: $[\mathbb{A}, \mathbb{B}] = \mathrm{i}\hbar$. Then (2) reduces to (1): $\Delta A \Delta B \geqslant \frac{1}{2}\hbar$†.

When $g < \infty$ there are no canonically conjugate operators (Wintner 1947, Wielandt 1949). In this case either lower bound in (2) is the mean value of some operator. Therefore they depend on the state $\mathbb{W}$. As pointed out by Judge (1963) and Davidson (1965), this represents a shortcoming of the uncertainty relations (2) for $g < \infty$, with

† Unbounded operators will not concern us here, and we refer elsewhere for a discussion of commutators and uncertainty relations in that case. All our relations in the present work apply within the scope of bounded operators, and to generalise will require attention to domain problems, etc.

respect to their characterisation of the aspects of $\mathbb{A}$ and $\mathbb{B}$. The following can happen: $\Delta A \neq 0$ and $\Delta B \neq 0$, but $|\langle \mathbb{A}\mathbb{B}\rangle - \langle \mathbb{A}\rangle\langle \mathbb{B}\rangle| = 0$. That is, $\mathbb{A}$ and $\mathbb{B}$ are incompatible, but (2) does not show it for those states $\mathbb{W}$ where the bound vanishes for an 'irrelevant' reason†.

For instance, for the Pauli operators $\Delta\sigma_x\Delta\sigma_y = \{(1-P_x^2)(1-P_y^2)\}^{1/2}$, and $=(1-P_x^2)^{1/2} > 0$ if $P_y = P_z = 0$ and $P_x < 1$ (mixed state). But here the bound is $\{P_z^2 + P_x^2 P_y^2\}^{1/2} = 0$, and we are not advised that σ_x and σ_y are incompatible.

1.2. Some recent results

Recent advances towards obtaining uncertainty relations which do not suffer from the above-mentioned shortcoming were initiated by Deutsch (1983). The idea in this development is to consider predictions (X, Y) other than the variances of two observables. Since the first alternative to be investigated was a pair of entropies, these new relations are currently referred to as 'entropic uncertainty relations'.

Suppose $\mathbb{A}$ and $\mathbb{B}$ belong to different aspects of the system: $\mathbb{A} \in \mathscr{A}_g$ and $\mathbb{B} \in \mathscr{A}_\ell$‡. In the state $\mathbb{W}$ the predicted outcomes of measuring either $\mathscr{A}_g$ or $\mathscr{A}_\ell$ are two sets of probabilities, $\{P(\alpha)\}$ and $\{P(n)\}$, where $P(\alpha) = \langle \mathbb{P}_\alpha\rangle = \langle\alpha|\mathbb{W}|\alpha\rangle$ and $P(n) = \langle\mathbb{P}_n\rangle = \langle n|\mathbb{W}|n\rangle$. If such complete measurements were actually carried out, the outcome would be systems in states§

$$\mathbb{w}_g = \sum_\alpha w_\alpha \mathbb{P}_\alpha \in \mathscr{A}_g \qquad w_\alpha = P(\alpha)$$

$$\mathbb{w}_\ell = \sum_n w_n \mathbb{P}_n \in \mathscr{A}_\ell \qquad w_n = P(n).$$

We can define either the Shannon information entropies for the predictions while the state is $\mathbb{W}$, or the Boltzmann-von Neumann entropy ($k_B = 1$) for either of the alternative states $\mathbb{w}_g$ or $\mathbb{w}_\ell$. For complete measurements these will give

$$H_g \equiv -\sum_\alpha P(\alpha)\ln P(\alpha) = -\mathrm{Tr}(\mathbb{w}_g \ln \mathbb{w}_g) \equiv S(\mathbb{w}_g) \tag{3}$$

$$H_\ell \equiv -\sum_n P(n)\ln P(n) = -\mathrm{Tr}(\mathbb{w} \ln \mathbb{w}_\ell) \equiv S(\mathbb{w}_\ell). \tag{4}$$

Deutsch (1983) showed that

$$H_g + H_\ell \geqslant -2\ln\left(\frac{1+c}{2}\right) \tag{5}$$

where

$$c \equiv \max_{\alpha,n} |\langle\alpha|n\rangle| \qquad \text{or} \qquad c^2 \equiv \max_{\alpha,n} \mathrm{Tr}(\mathbb{P}_\alpha\mathbb{P}_n). \tag{6}$$

Proving a conjecture by Kraus (1987), Maassen and Uffink (1988) very recently sharpened (5) to

$$H_g + H_\ell \geqslant -2\ln(c). \tag{7}$$

† Of course, the bound in (2) may vanish for a relevant reason: $\mathbb{W}$ may be an eigenstate, of $\mathbb{A}$ say, so that $\Delta A = 0$. But this must be a point on the $(X, Y) = (\Delta A, \Delta B)$ boundary, and so it is not a shortcoming.

‡ Here $\mathscr{A}_g \equiv \{\mathbb{A}|\mathbb{A} = \Sigma_\alpha A_\alpha\mathbb{P}_\alpha,\ \mathbb{P}_\alpha = |\alpha\rangle\langle\alpha|\}$, and $\mathscr{A}_\ell \equiv \{\mathbb{B}|\mathbb{B} = \Sigma_n B_n\mathbb{P}_n,\ \mathbb{P}_n = |n\rangle\langle n|\}$ for a 'Greek' basis $\{|\alpha\rangle\}$ and a 'Latin' basis $\{|n\rangle\}$ in $\mathbb{H}$. There are infinitely many such 'aspects' of even $g = 2$ systems (cf Larsen 1988).

§ Complete measurements were discussed in detail in Larsen (1986b, 1988), where it was shown that the agreement $w_\alpha = P(\alpha)$ between a measured statistic $\{w_\alpha\}$ and predicted probabilities $\{P(\alpha)\}$ need not be postulated, but can be obtained as a theorem on very plausible measurement axioms.

This bound is independent of the initial state $\mathbb{W}$, which in (7), like in (2), can be any pure or mixed state. Via the overlap c, the bound therefore properly displays the incompatibility between a pair of non-intersecting aspects†. The entropy is a general measure of the dispersion to be expected if a measurement of, say, $\mathbb{A} \in \mathscr{A}_g$ were to be performed—as well as of the actual statistical dispersion present in the resulting state $\mathbb{w}_g$, after the measurement. The entropic measure applies to all member operators of $\mathscr{A}_g$, and is more informative than any single variance.

Hall and Santhanam (1989) consider the incompatibility measure

$$I(\mathscr{A}_g, \mathscr{A}_\ell, \mathbb{W}) \equiv \max_{\alpha,n} \mathrm{Tr}(\mathbb{W}(\mathbb{P}_\alpha \vee \mathbb{P}_n)) \tag{8}$$

relative to a given state $\mathbb{W}$. Here $\mathbb{P}_\alpha \vee \mathbb{P}_n$ is the projector on the subspace of $\mathbb{H}$ spanned by $|\alpha\rangle$ and $|n\rangle$. They show

$$I(\mathscr{A}_g, \mathscr{A}_\ell, \mathbb{W}) \geqslant \frac{1}{g}\frac{2}{1+c} \qquad \text{if } [\mathbb{P}_\alpha, \mathbb{P}_n] \neq 0 \text{ for all } \alpha, n. \tag{9}$$

Maassen and Uffink (1988) also consider a more general class of measures of dispersion. For $r, s \geqslant -1$ these measures are defined as

$$M_r(\mathbb{w}_g) = \left\{\sum_\alpha w_\alpha^{1+r}\right\}^{1/r} \tag{10}$$

$$M_s(\mathbb{w}_\ell) = \left\{\sum_n w_n^{1+s}\right\}^{1/s} \tag{11}$$

and are essentially the class of p-norms for compact operators: $\|\mathbb{A}\|_p = \{\mathrm{Tr}(|\mathbb{A}|^p)\}^{1/p}$, where $|\mathbb{A}|^2 = \mathbb{A}^+\mathbb{A}$ (Dunford and Schwartz 1963). They find

$$M_r(\mathbb{w}_g)M_s(\mathbb{w}_\ell) \leqslant c^2 \qquad \begin{array}{ll} \text{for } r = -s/(2s+1) & \text{if } s \geqslant 0 \\ \text{and } s = -r/(2r+1) & \text{if } r \geqslant 0. \end{array} \tag{12}$$

Equation (7) corresponds to the special case $r = s = 0$. For other values of (r, s) (12) may be a sharper bound than (7), but this requires that r and s have opposite signs‡.

These new bounds have many desirable features, most importantly the independence of $\mathbb{W}$. But it has not been demonstrated that these bounds are the best possible, in the sense of being always attainable, or of c^2 being a supremum in (12), and $-2\ln(c)$ an infimum in (7). However, if $\mathscr{A}_g$ and $\mathscr{A}_\ell$ are in the relation where $|\langle\alpha|n\rangle|^2 = 1/g$ for all α, n (cf section 2), then $-2\ln(c) = \ln(g)$, and (7) is optimal. For pure states $\mathbb{W}$ we may have $\mathbb{W} = \mathbb{w}_g \in \mathscr{A}_g$ and get $\mathbb{w}_\ell = (1/g)\mathbb{1}$, giving $H_g = 0$ and $H_\ell = \ln(g)$. Besides this special case no claim of attainability is made.

A shortcoming (for instance pointed out by Landau and Pollak (1961), Lenard (1972) and Kantor (1986)) of such uncertainty relations—as distinct from the exhaustive 'uncertainty relationship'—is their commitment to a specific functional form. The relations (7) or (12) cannot be counted upon to coincide with the boundary of the allowed region of the pair of variables concerned. A complete display of the exact uncertainty relationship, which is the aim of the present investigation, requires that the boundary itself be located.

† The bound $-2\ln(c)$ can only equal zero if there is at least one $|\langle\alpha|n\rangle| = 1$, and then it must, because in that case there is a state $\mathbb{W} = \mathbb{P}_\alpha = \mathbb{P}_n = \mathbb{w}_g = \mathbb{w}_\ell$ for which $H_g = H_\ell = 0$. The non-intersection relationship between aspects will be precisely defined in the following sections.

‡ Incidentally, in terms of the p-norms, for $p \geqslant 1$ the inequality (12) reads: $\|\mathbb{w}_g\|_p \leqslant (c^2)^{1-1/p}\|\mathbb{w}_\ell\|_q$, where $1/p + 1/q = 2$.

It is beyond the scope of the present work to discuss the uncertainty relationship pertaining to such continuous variables as the q and p which enter the Heisenberg relation (1). The corresponding operators do not belong to aspects (cf Larsen (1988) for a discussion), but may be arbitrarily well approximated by ones that do, as $g \to \infty$. The wave-particle dual representations, $\psi(q) = (q|\psi\rangle$ and $\tilde{\psi}(p) = (p|\psi\rangle$, properly belong to Fourier theory. For the associated probability densities, $|\psi(q)|^2$ and $|\tilde{\psi}(p)|^2$, Hirschmann (1957) and Beckner (1975) provided the first 'entropic uncertainty relation', which was used by Bialynicki-Birula and Mycielski (1975) to obtain inequalities which combine the variances of (1) with the information entropies for $|\psi(q)|^2$ and $|\tilde{\psi}(p)|^2$.

1.3. *Some new uncertainty relations*

In the present work we consider the Hilbert–Schmidt norm $\|\cdot\|_2$ as dispersion measure, for both final states $\mathbb{w}_g$ and $\mathbb{w}_\ell$, as well as for the initial state $\mathbb{W}$†. The best variables are the purities

$$
\begin{aligned}
&\Pi \equiv \mathrm{Tr}(\mathbb{W}^2) && 0 \leqslant \bar{\Pi} = \Pi - 1/g \leqslant 1 - 1/g \\
&\pi_g \equiv \mathrm{Tr}(\mathbb{w}_g^2) = M_1(\mathbb{w}_g) && \bar{\pi}_g \equiv \pi_g - 1/g \\
&\pi_\ell \equiv \mathrm{Tr}(\mathbb{w}_\ell^2) = M_1(\mathbb{w}_\ell) && \bar{\pi}_\ell \equiv \pi_\ell - 1/g
\end{aligned}
\tag{13}
$$

where the first set refers to $\mathscr{B}_2$, and the second set to $\bar{\mathscr{B}}_2$ (cf appendix for definitions). The superspace $\mathscr{B}_2$ and its hyperplane $\bar{\mathscr{B}}_2$ of traceless operators have Euclidean geometry, due to the scalar product $\mathrm{Tr}(\mathbb{A}^\dagger \mathbb{B})$. Thus $1/g \leqslant \Pi = \|\mathbb{W}\|_2^2 \leqslant 1$, and $\mathbb{W}$ is pure iff $\Pi = 1$, or $\bar{\Pi} = 1 - 1/g$. Apart from the advantage of using the same dispersion measure for both aspects $\mathscr{A}_g$ and $\mathscr{A}_\ell$, it is straightforward to extend the present results to $g = \infty$‡.

With this geometry in $\mathscr{B}_2$ and $\bar{\mathscr{B}}_2$, finding the allowed region of the pair (π_g, π_ℓ), or $(\bar{\pi}_g, \bar{\pi}_\ell)$, reduces to showing that Lenard's (1972) theorem on the numerical range of a pair of projectors applies. This is straightforward, and we give the result in section 3.4. Also, because Lenard's theorem relies on the 'two-subspaces' theorem of Halmos (1969) which may not be familar to all physicists, we provide an independent proof using only elementary geometry in $\bar{\mathscr{B}}_2$ (section 3.5).

For the purpose of comparing with earlier uncertainty relations, from the general result we may deduce the following relations§. Let ϕ_m be the minimal angle between two aspects, $\mathscr{A}_g$ and $\mathscr{A}_\ell$, for which an exact expression is given in section 3.1. Then‖

$$\bar{\pi}_g + \bar{\pi}_\ell \leqslant \bar{\Pi}(1 + \cos\phi_m) \qquad 0 \leqslant \phi_m \leqslant \pi/2. \tag{14}$$

Equality is attainable, if one starts with states $\mathbb{W}$ of small enough purity $\bar{\Pi}$. However, since the pure states do not in general cover the whole 'unit' sphere in $\bar{\mathscr{B}}_2$ (cf Bloore

† This corresponds to the case $r = s = 1$ in which (12) does not apply.

‡ As pointed out by Maassen and Uffink (1988), this is not so for one of the measures used in (12). The reason is this: consider the nested classes $\mathscr{B}_p$ of p-compact operators (cf Dunford and Schwartz 1963). Here $\mathscr{B}_p \subseteq \mathscr{B}_{p'}$ if $p \leqslant p'$. One of the measures M_r or M_s in (12) corresponds to a norm $\|\cdot\|_p$ with $p < 1$. But a state, e.g. $\mathbb{w}_g$, is only required to be in $\mathscr{B}_1$, and so for $g \to \infty$ $\|\mathbb{w}_g\|_p \to \infty$ is imminent. If that happens, (12) cannot be extended. On the other hand, since $\mathscr{B}_1 \subseteq \mathscr{B}_2$ no similar problem arises with the purities of (13).

§ Since the latter are bounds, like (7) and (12), a variety of functional forms may be chosen. The ones we consider touch the boundary at the point where $\pi_g = \pi_\ell$.

‖ If there are states in common between $\mathscr{A}_g$ and $\mathscr{A}_\ell$ (apart from $\mathbb{W}_0 = (1/g)\mathbb{1}$), then $\phi_m = 0$; if $\mathscr{A}_g$ and $\mathscr{A}_\ell$ are in the special configuration mentioned in section 1.2, then $\phi_m = \pi/2$ (maximal incompatibility).

1976), equality may not be attainable for the outer inequality obtained from (14) setting $\bar{\Pi} = 1 - 1/g$. But with respect to displaying the incompatibility between the two aspects this is of no consequence, since $\bar{\Pi}$ does not depend on $\mathscr{A}_g$ or $\mathscr{A}_\ell$. Equation (14) adequately exhibits the geometrical nature of the relationship between the two aspects, which we shall discuss in detail in section 2.

Another uncertainty relation, not quite as sharp as (14), is of interest for comparison with (12). It is

$$\pi_g \pi_\ell = M_1(\mathbb{W}_g) M_1(\mathbb{W}_\ell) \leqslant \left[\tfrac{1}{2}\bar{\Pi}(1 + \cos\phi_m) + \frac{1}{g} \right]^2. \tag{15}$$

Either of these relations shows how (π_g, π_ℓ) is confined away from the point (Π, Π), which can only be reached if $\phi_{\mathrm{m}} = 0$. That is, if $\mathscr{A}_g$ and $\mathscr{A}_\ell$ are not entirely incompatible.

Naturally the overlap c^2 does not enter these expressions—the angle ϕ_{m} is a more precise characterisation. An exception is the case of total incompatibility, where $c^2 = 1/g$ and $\phi_{\mathrm{m}} = \pi/2$. For instance, letting $\Pi = 1$, (15) can be written as $\pi_g \pi_\ell \leqslant c^2 + \frac{1}{4}(1 - c^2)^2$, and (14) as $\pi_g + \pi_\ell \leqslant 1 + c^2$.

Apart from the exact expression for ϕ_{m}, in section 3.6 we show that a bound (not a sharp one) on '$\cos\phi_{\mathrm{m}}$' can be obtained in terms of c^2, valid for $\Pi \geqslant c^2$. It is ($\phi'_{\mathrm{m}} \equiv$ minimum angle for $\mathbb{W}$ of fixed Π)

$$\cos\phi'_{\mathrm{m}} \leqslant \frac{1}{2}\left\{ \left[1 + 8\frac{c^2 - 1/g}{\Pi - 1/g} \right]^{1/2} - 1 \right\} \qquad c^2 \leqslant \Pi \leqslant 1. \tag{16}$$

But in many cases the standard bounds obtainable from matrix theory may be better (cf section 3.3).

A more extensive uncertainty relation is proved in section 2.4, which relates the purities of a set of $g + 1$ totally incompatible aspects. This relation, (25), is significant in connection with the precise characterisation of complementarity.

2. Complementary aspects

2.1. Measuring aspects

Let $\mathbb{H}$ be a g-dimensional Hilbert space, and $\{|\alpha\rangle \in \mathbb{H} | \langle\alpha|\beta\rangle = \delta_{\alpha\beta}\}$ an orthonormal basis. With $\mathscr{L}_g \equiv \{\mathbb{P}_\alpha = |\alpha\rangle\langle\alpha| \,|\, \mathbb{P}_\alpha \mathbb{P}_\beta = \mathbb{P}_\alpha \delta_{\alpha\beta}\}$ the corresponding aspect is the set

$$\mathscr{A}_g = \left\{ \mathbb{A} = \sum_\alpha A_\alpha \mathbb{P}_\alpha \,\middle|\, \mathbb{P}_\alpha \in \mathscr{L}_g, A_\alpha \in \mathbb{C} \right\} \tag{17}$$

consisting of all operators which can be simultaneously diagonalised in terms of $\mathscr{L}_g$.

The set of states is

$$\mathscr{S} = \{ \mathbb{W} | \mathbb{W} > 0, \mathrm{Tr}(\mathbb{W}) = 1 \}. \tag{18}$$

According to the Hilbert-Riesz theorem (cf Conway 1985, Dunford and Schwartz 1963, etc) any $\mathbb{W} \in \mathscr{S}$ belongs to an aspect: e.g. $\mathbb{W} = \Sigma_{a=1}^{g} W_a \mathbb{P}_a$, where $\mathscr{L} = \{\mathbb{P}_a\}$ defines some aspect $\mathscr{A}$.

If aspects to which $\mathbb{W}$ does not belong are to be investigated measurements must be performed, or considered theoretically. A measurement of an aspect, say $\mathscr{A}_g$, is a transformation $T_g: \mathscr{S} \to \mathscr{S}$. We require (for further details cf Larsen 1988) that, (i) the final state belongs to $\mathscr{A}_g$

$$\mathbb{W}_g = T_g(\mathbb{W}) = \sum_\alpha w_\alpha \mathbb{P}_\alpha \in \mathscr{A}_g \tag{19}$$

and (ii) T_g leaves all pure states in $\mathscr{L}_g$ invariant: $T_g(\mathbb{P}_\alpha) = \mathbb{P}_\alpha$, for all $\alpha = 1, \ldots, g$.

These axioms suffice to show that any such T_g projects orthogonally on $\mathscr{A}_g$, considering transformations as positive linear operators on the superspace $\mathscr{B}_2$ in which $\mathscr{S}$ is embedded (cf appendix). Furthermore

$$w_\alpha = \langle \alpha | \mathbb{W} | \alpha \rangle \equiv P(\alpha) \tag{20}$$

where $\{P(\alpha)\}$ are the predicted probabilities for $\mathbb{W}$ with respect to $\mathscr{A}_g$.

For example, T_g may be the von Neumann (1955) projection

$$T_g(\cdot) = \sum_\alpha \mathbb{P}_\alpha \cdot \mathbb{P}_\alpha . \tag{21}$$

Note that, with our axioms (i) and (ii) one need not postulate (21) in order to define measurements. It becomes a member of a class $\mathscr{M}(\mathscr{A}_g)$ of transformations which all satisfy (19) and (20), and in which one also finds transformations with a physically meaningful causal structure.

The preceding definitions apply to every aspect, such as our alternative $\mathscr{A}_\ell$.

2.2. *Complementary aspects are necessary*

Every transformation, such as T_g, has an objective causal representation (Larsen 1986a, 1988 and references therein). Thus

$$T_g(\cdot) = \sum_j \mathbb{E}_j \cdot \mathbb{E}_j^\dagger \tag{22}$$

where the set $\mathscr{C} = \{\mathbb{E}_j \in \mathscr{B}(\mathbb{H})\}$ is referred to as the 'objective cause' of the transformation. In (21) $\mathscr{C} = \mathscr{L}_g$, for example.

Another consequence of axioms (i) and (ii) is that, for all members of $\mathscr{M}(\mathscr{A}_g)$ the measurement requires a cause $\mathscr{C}$ whose operators $\mathbb{E}_j$ all belong to $\mathscr{A}_g$ itself. Physically this means that the operators in an aspect define the very operations needed to perform their own measurement.

The incompatibility of different aspects, such as $\mathscr{A}_g$ and $\mathscr{A}_\ell$, refers to the fact that the operations needed to measure $\mathscr{A}_g$ or $\mathscr{A}_\ell$ cannot be carried out simultaneously (not all operators in $\mathscr{A}_g$ and $\mathscr{A}_\ell$ commute). A full identification of an (unknown) initial state $\mathbb{W}$ requires that a set of replica systems are prepared, all in $\mathbb{W}$, whereupon a set of different aspects can be measured. Each measurement provides a 'projected image', such as $\mathbb{W}_g = T_g(\mathbb{W})$ when $\mathscr{A}_g$ is measured†.

Roughly speaking, the information content of a state $\mathbb{W}$ resides in $g^2 - 1$ real numbers. A projected image on any aspect provides *at most* $g - 1$ real numbers $\in \{w_\alpha | 0 \leq w_\alpha \leq 1, \Sigma_\alpha w_\alpha = 1\}$. So it requires *at least* $n = g + 1$ images projected on as many different aspects in order to fully identify $\mathbb{W}$.

† For example, in the binary system ($g = 2$) one has $\mathbb{W} = \frac{1}{2}(\mathbb{1} + \vec{P} \cdot \vec{\sigma})$, where $\vec{P} = \langle \vec{\sigma} \rangle$ is the polarisation (in $\mathbb{R}^3$). Each direction of polarisation defines an aspect, e.g. $\sigma_z = \mathbb{P}_+ - \mathbb{P}_- \in \mathscr{A}_z$ defined by $\mathscr{L}_z = \{\mathbb{P}_+, \mathbb{P}_-\}$. A measurement of $\mathscr{A}_z$ is a projection on the z axis in $\mathbb{R}^3$: $\mathbb{W}_z = T_z(\mathbb{W}) = \frac{1}{2}(\mathbb{1} + \vec{P}_z \cdot \vec{\sigma})$, where $\vec{P}_z = (0, 0, \hat{z} \cdot \vec{P})$. Thus to identify $\mathbb{W}$ it takes three projections of $\vec{P}$ on three linearly independent $\mathbb{R}^3$ directions, e.g. $\vec{P} = (P_x, P_y, P_z)$ for the aspects $\mathscr{A}_x$, $\mathscr{A}_y$, $\mathscr{A}_z$. This can be done by means of incompatible rotations about the x, y and z axes, generated by σ_x, σ_y, and σ_z, respectively (further details in Larsen 1988).

Aspects are subspaces of $\mathscr{B}_2$ (cf appendix). Let the linear span of two aspects $\bigvee\{\mathscr{A}_1, \mathscr{A}_2\}$ be the set of all operators which can be expressed as a linear combination of elements from $\mathscr{A}_1$ and $\mathscr{A}_2$. For n different aspects

$$g + n - 1 \leq \dim(\bigvee\{\mathscr{A}_1, \ldots, \mathscr{A}_n\}) \leq n(g-1) + 1. \tag{23}$$

In order to span all of $\mathscr{B}_2$ we need $g + 1 \leq n \leq g^2 - g + 1$, since $\dim(\mathscr{B}_2) = g^2$.

The standard concept of complementarity (Bohr 1963) thus implies the following definition.

Definition (objective complementarity). $\mathscr{C}mp = \{\mathscr{A}_\mu | 1 \leq \mu \leq n\}$ is a set of complementary aspects provided† $\bigvee\{\mathscr{A}_\mu\} \cap \mathscr{B}_2 = \mathscr{B}_2$, and provided there is no redundancy, i.e. provided every proper subset of $\mathscr{C}mp$ spans a proper subspace of $\mathscr{B}_2$.

This definition emphasises the completeness inherent in the original concept—that a full identification of an arbitrary (unknown) object state $\mathbb{W}$ is possible by means of (complete) measurements of a set of complementary aspects. That this is so follows from the geometrical significance of the projections $T_\mu \in \mathscr{M}(\mathscr{A}_\mu)$, because $\mathscr{C}mp$ contains a basis for $\mathscr{B}_2$.

It should be remarked that it is the very existence of options to perform incompatible transformations (non-commuting operators) which demands that a full empirical identification of the state of an object must take place through complementary pictures—pictures obtained under the mutually exclusive conditions which causally define (through $\mathscr{C}$) the corresponding measurement transformations. According to our definition, a set $\mathscr{C}mp$ is one which just suffices for this purpose.

2.3. *Perpendicular aspects*

There are many ways to compose $\mathscr{C}mp$ sets of complementary aspects. Evidently the questions with respect to aspects are more complicated than those pertaining to a quorum, about which more is known (cf appendix). We ask not just for a basis in $\mathscr{B}_2$, but for one which as far as possible consists of subsets of mutually compatible properties. Nevertheless, aspects are what is measured (cf Wootters 1986a). A quorum would generally involve more aspects than what is strictly necessary in order to form a $\mathscr{C}mp$ set.

Although the question of precisely which aspects constitute such a set $\mathscr{C}mp$ must have been asked frequently in the past, the only exact results of which we are aware have been obtained quite recently (Ivanović 1981, 1983).

For the cases where g is a prime number Ivanović (1981) has shown by explicit construction that there exist sets $\mathscr{C}mp = \{\mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_g\}$ constituted of $g+1$ aspects whose components (cf appendix) in $\bar{\mathscr{B}}_2$ are mutually orthogonal: $\mathrm{Tr}(\bar{\mathbb{A}}^{(\mu)}\bar{\mathbb{B}}^{(\mu')}) = 0$ for any such pair of aspects $\mathscr{A}_\mu$ and $\mathscr{A}_{\mu'}$ $(\mu \neq \mu')$ in $\mathscr{C}mp$. We refer to such aspects as perpendicular‡.

† If $g = \infty$ then the aspects may contain operators outside $\mathscr{B}_2$, but since $\mathscr{S} \subset \mathscr{B}_2$, and since only $\mathscr{B}_2$ has Euclidean geometry, being the only Hilbert space among $\mathscr{B}_p$ and $\mathscr{B}(\mathbb{H})$, it is $\mathscr{B}_2$ that counts.

‡ Some authors refer to what we call perpendicular aspects as 'complementary' aspects (e.g. Schwinger 1960b, Kraus 1987). However, in these works only pairs of aspects are considered, such as Schwinger's Fourier-dual 'wave-particle' aspects. A pair of perpendicular aspects always exists, but to incorporate the completeness property of complementarity we need a generalisation of Ivanović's result to non-prime g. It is not known if this is possible. If not, then resort must be taken to less optimal sets $\mathscr{C}mp$.

In this optimal situation it is straighforward to reconstruct the original state $\mathbb{W}$ out of the $g+1$ states $\mathbb{w}^{(\mu)}=T_\mu(\mathbb{W})\in\mathscr{A}_\mu$ which are the results of measuring $\mathscr{C}mp$ by means of transformations $T_\mu\in\mathscr{M}(\mathscr{A}_\mu)$ on $g+1$ replicas of $\mathbb{W}$. Restricted to $\bar{\mathscr{B}}_2$ these measurements $\{T_\mu|\mu=0,1,\ldots,g\}$ are projectors on $g+1$ orthogonal $(g-1)$-dimensional subspaces spanning $\bar{\mathscr{B}}_2$: $\bar{\mathscr{B}}_2=\bigoplus_{\mu=0}^{g}\bar{\mathscr{A}}_\mu$.

Let $\bar{\mathbb{W}}=\mathbb{W}-(1/g)\mathbb{1}\in\bar{\mathscr{S}}=\mathscr{S}\cap\bar{\mathscr{B}}_2$. Then $\Sigma_{\mu=0}^{g}T_\mu(\bar{\mathbb{W}})=\bar{\mathbb{W}}$, since $\Sigma_{\mu=0}^{g}T_\mu(\cdot)=$ unit operator on $\bar{\mathscr{B}}_2$. Therefore

$$\mathbb{W}=\sum_{\mu=0}^{g}T_\mu(\mathbb{W})-\mathbb{1}=\sum_{\mu=0}^{g}\mathbb{w}^{(\mu)}-\mathbb{1} \tag{24}$$

(Ivanović 1981). Clearly a similar analysis of the data obtained from measurements of less perfectly arranged complementary aspects would also permit to identify $\mathbb{W}$.

It remains an open question if sets $\mathscr{C}mp$ composed exclusively of mutually perpendicular aspects exist when g is not prime. Some later results of Ivanović (1983) pertain to sets of aspects whose components in $\bar{\mathscr{B}}_2$ are not orthogonal, but linearly independent†. For $g=\infty$ the discussion in section 2.2 establishes the existence of $\mathscr{C}mp$ sets—not much more seems to be known at present. Of course, if taking $g\to\infty$ in the present $g<\infty$ discussion implies that every $\mathscr{C}mp$ set becomes infinite, then a full identification of an arbitrary $\mathbb{W}$ becomes unfeasible, by empirical means. Hence the traditional emphasis on the investigation of suitably prepared and reproducible initial states—*vis-à-vis* spontaneously arising ones. In our opinion, such practical difficulties should not be taken to affect the physical reality which is associated with the state of a system (cf section 4).

2.4. A master uncertainty relation

First, suppose $\mathbb{W}\in\mathscr{A}_g$, where $\mathscr{A}_g$ is the last of the $g+1$ perpendicular aspects listed in $\mathscr{C}mp=\{\mathscr{A}_0,\mathscr{A}_1,\ldots,\mathscr{A}_g\}$. Then for all $\mu=0,\ldots,g-1$: $T_\mu(\bar{\mathbb{W}})=T_\mu(\mathbb{W}-(1/g)\mathbb{1})=0$, i.e. $\mathbb{w}^{(\mu)}=(1/g)\mathbb{1}=\mathbb{W}_0$. The result of measuring any aspect which is perpendicular to the aspect of the initial state is the completely mixed state $\mathbb{W}_0$. The data obtained, i.e. $\{w_{\pi_\mu}^{(\mu)}|\mu=0,\ldots,g\}$, agree with the predicted probabilities $\{P^{(\mu)}(n_\mu)\}$. So obtaining $\mathbb{W}_0$ implies that all properties in aspects perpendicular to $(\mathscr{A}_g)$ are totally indeterminate, when $\mathbb{W}\in\mathscr{A}_g$, whether $\mathbb{W}$ is pure or not. The uncertainties predicted are maximal, and a measurement of any aspect perpendicular to $\mathscr{A}_g$ provides no information at all about the values of the properties in it‡.

More generally, when $\mathbb{W}$ does not belong to any of the aspects in $\mathscr{C}mp$, let $\Pi=\mathrm{Tr}(\mathbb{W}^2)$ and $\pi^{(\mu)}\equiv\mathrm{Tr}(\mathbb{w}^{(\mu)2})$ be purities. Since $\bar{\mathbb{w}}^{(\mu)}=\mathbb{w}^{(\mu)}-(1/g)\mathbb{1}$ belong to orthogonal subspaces $\bar{\mathscr{A}}_\mu$ in $\bar{\mathscr{B}}_2$, Pythagoras' theorem on $\bar{\mathbb{W}}=\Sigma_{\mu=0}^{g}\bar{\mathbb{w}}^{(\mu)}$ gives $0<\Sigma_{\mu=0}^{g}\mathrm{Tr}(\bar{\mathbb{w}}^{(\mu)2})=\mathrm{Tr}(\bar{\mathbb{W}}^2)=\bar{\Pi}=\Pi-(1/g)\leqq 1-(1/g)$. Thus

$$1+\frac{1}{g}\leqslant\sum_{\mu=0}^{g}\pi^{(\mu)}=\Pi+1\leqslant 2 \qquad (\text{all } \mathscr{A}_\mu\perp). \tag{25}$$

The upper bound is attained when $\mathbb{W}$ is pure, and the lower bound when $\mathbb{W}=\mathbb{W}_0$. In particular, for a pair of perpendicular aspects $\mathscr{A}_\varphi,\mathscr{A}_\ell\in\mathscr{C}mp$ we have (strictly

† Linear independence is related to projectors T in 'generic position'. This will be discussed in section 3.
‡ Such a measurement, of course, *does* provide information about $\mathbb{W}$ in the sense of section 1. But all values of physical properties come out in equal proportions, so the measurement says nothing about *them*.

speaking, only for prime g, so far) $\pi_g + \pi_\ell \leqslant \Pi + 1/g$, or in terms of $\bar{\mathscr{B}}_2$ norms

$$\bar{\pi}_g + \bar{\pi}_\ell \leqslant \bar{\Pi} \qquad (\mathscr{A}_g \perp \mathscr{A}_\ell). \tag{26}$$

This corresponds to (14) when $\phi_m = \pi/2$.

For example, for $g = 2$ with $\mathscr{C}m\not{p} = \{\mathscr{A}_x, \mathscr{A}_y, \mathscr{A}_z\}$ we obtain the uncertainty relation for variances ($\Delta\sigma_x^2 = 2(1 - \pi^{(x)})$, etc)

$$2 \leqslant \Delta\sigma_x^2 + \Delta\sigma_y^2 + \Delta\sigma_z^2 = 3 - P^2 \leqslant 3. \tag{27}$$

Although the Euclidean geometry make these relations straightforward to derive, they are in fact stronger statements than the traditional relations, such as (2).

3. The exact uncertainty relationship

3.1. The angle between two aspects

It is convenient to use ordinary vector notation for the $\bar{\mathscr{B}}_2$ geometry: $\bar{\mathbb{W}} = \boldsymbol{W}$, $\bar{\mathbb{w}}_g = \boldsymbol{x}$, and $\bar{\mathbb{w}}_\ell = \boldsymbol{y}$. Thus the scalar product is

$$\boldsymbol{x} \cdot \boldsymbol{y} = \mathrm{Tr}(\bar{\mathbb{w}}_g \bar{\mathbb{w}}_\ell) \tag{28}$$

and the $(g^2 - 1)$-dimensional vectors have lengths

$$x \equiv |\boldsymbol{x}| = \|\bar{\mathbb{w}}_g\|_2 = \{\mathrm{Tr}(\bar{\mathbb{w}}_g^2)\}^{1/2} = \left\{\pi_g - \frac{1}{g}\right\}^{1/2} = \sqrt{\bar{\pi}_g}. \tag{29}$$

Also, the angle ϕ between $\bar{\mathbb{w}}_g$ and $\bar{\mathbb{w}}_\ell$ is

$$\cos\phi = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{xy} = \mathrm{Tr}(\bar{\mathbb{w}}_g \bar{\mathbb{w}}_\ell) / \|\bar{\mathbb{w}}_g\|_2 \|\bar{\mathbb{w}}_\ell\|_2. \tag{30}$$

A measurement of $\mathscr{A}_g$ projects orthogonally on $\mathscr{A}_g$: $\mathbb{w}_g = T_g(\mathbb{W})$. In $\bar{\mathscr{B}}_2$ on $\bar{\mathscr{A}}_g$ (cf appendix): $\boldsymbol{x} = T_g(\boldsymbol{W})$. Likewise: $\boldsymbol{y} = T_\ell(\boldsymbol{W})$. The distance from any $\boldsymbol{W}$ to $\bar{\mathscr{A}}_g$ is shortest between $\boldsymbol{W}$ and the projected image $\boldsymbol{x}$ (true in all Hilbert spaces).

Any two aspects $\bar{\mathscr{A}}_g$ and $\bar{\mathscr{A}}_\ell$ intersect at the origin of $\bar{\mathscr{B}}_2$. If the two aspects are linearly independent (Ivanović 1983) the origin O is the only point of intersection. In that case the corresponding projectors, T_g and T_ℓ, restricted to $\bar{\mathscr{B}}_2$, are said to be in generic position (Halmos 1969). If $\bar{\mathscr{A}}_g$ and $\bar{\mathscr{A}}_\ell$ intersect elsewhere than in O, then a subset of states $\boldsymbol{W}$ exists, for which $T_g(\boldsymbol{W}) = T_\ell(\boldsymbol{W}) = \boldsymbol{W}$. Hence there is no uncertainty relation between x and y, as all that can be said is $0 \leqslant \phi \leqslant \pi/2$. On the other hand, if $\bar{\mathscr{A}}_g$ and $\bar{\mathscr{A}}_\ell$ are perpendicular (cf appendix), then $\phi = \pi/2$ for all $\boldsymbol{x} \in \bar{\mathscr{A}}_g$ and $\boldsymbol{y} \in \bar{\mathscr{A}}_\ell$.

In order to define the minimal angle, ϕ_m, and the maximal angle, ϕ_M, between a given pair of aspects, consider the unit spheres in $\bar{\mathscr{A}}_g$ and $\bar{\mathscr{A}}_\ell$, which we assume are linearly independent. These spheres do not intersect. The shortest distance between them is the length of one or more vectors lying in a plane through O which intersects both spheres at right angles.

To see this, assume it is not the case. Project perpendicularly from one unit-sphere point onto the other's aspect. This will reach a point perpendicularly opposite, and at the shortest distance. The angle from O to these two points is the smallest angle between the initial unit vector and any vector in the aspect projected upon. This angle is smaller than the angle between the two initial unit vectors. Thus we have a contradiction.

The unit ball in $\bar{\mathscr{A}}_g$, say, maps one-to-one onto the simplex $\bar{\mathscr{S}}_g$ (Brown and Wehrl 1972). For every possible angle between pairs of vectors in $\bar{\mathscr{A}}_g$ and $\bar{\mathscr{A}}_\ell$ there are states, members of $\bar{\mathscr{S}}_g$ and $\bar{\mathscr{S}}_\ell$, respectively, which form the same angle, but possibly not pure states. Furthermore, there are states $\boldsymbol{W}$ in the plane which contains ϕ_{m} for which the projections T_g and T_ℓ both take place within said plane, so that ϕ_{m} is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$. Consequently

$$\cos^2 \phi_{\mathrm{m}} = \text{maximal eigenvalue of } T_g T_\ell | \bar{\mathscr{A}}_g \tag{31}$$

the restriction of the operator $T_g T_\ell$ to $\bar{\mathscr{A}}_g$. For the associated eigenvector, call it $\boldsymbol{x}_{\mathrm{m}}$, we get $T_g T_\ell(\boldsymbol{x}_{\mathrm{m}}) = T_g(\cos\phi_{\mathrm{m}}\boldsymbol{y}_{\mathrm{m}}) = \cos^2\phi_{\mathrm{m}}\boldsymbol{x}_{\mathrm{m}}$†. This agrees with Lenard's (1972) deduction from Halmos' (1969) two-subspaces theorem, the two subspaces in question being $\bar{\mathscr{A}}_g$ and $\bar{\mathscr{A}}_\ell$.

The same plane which contains ϕ_{m} also contains $\phi_{\mathrm{M}}^{\perp}$, the largest angle between, say, $\bar{\mathscr{A}}_g$ and $\bar{\mathscr{A}}_\ell^{\perp} \equiv \bar{\mathscr{B}}_2 \ominus \bar{\mathscr{A}}_\ell$ (the orthocomplement of $\bar{\mathscr{A}}_\ell$ in $\bar{\mathscr{B}}_2$). Were this not the case, then the plane which did contain a larger $\phi^{\perp}$ would have a complement angle $\phi > \phi_{\mathrm{m}}$, while $\phi^{\perp} + \phi = \pi/2$ implies $\phi^{\perp} = \pi/2 - \phi < \pi/2 - \phi_{\mathrm{m}} = \phi_{\mathrm{M}}^{\perp}$, a contradiction.

Inverting the roles of $\bar{\mathscr{A}}_\ell$ and $\bar{\mathscr{A}}_\ell^{\perp}$, this shows that the largest angle, ϕ_{M}, between $\bar{\mathscr{A}}_g$ and $\bar{\mathscr{A}}_\ell$ equals $\pi/2 - \phi_{\mathrm{m}}^{\perp}$, where $\phi_{\mathrm{m}}^{\perp}$ is the smallest angle between $\bar{\mathscr{A}}_g$ and $\bar{\mathscr{A}}_\ell^{\perp}$. Thus

$$\sin^2 \phi_{\mathrm{M}} = \cos^2 \phi_{\mathrm{m}}^{\perp} = \text{maximal eigenvalue of } T_g T_\ell^{\perp} | \bar{\mathscr{A}}_g \tag{32}$$

where $T_\ell^{\perp}$ is the orthogonal projector on $\bar{\mathscr{A}}_\ell^{\perp}$. This also agrees with the findings of Lenard (1972). The following are new results.

In order to find explicit expressions for ϕ_{m} and ϕ_{M} we employ a matrix representation on $\mathscr{A}_g$ (not $\bar{\mathscr{A}}_g$, which is inconvenient). An orthonormal basis for $\mathscr{A}_g$ is $\mathscr{L}_g = \{\mathbb{P}_\alpha\}$. To represent the restrictions to $\bar{\mathscr{A}}_g$ we need the projector P_0 on the $\mathscr{B}_2$ 'unit-subspace' $\bigvee\{\mathbb{1}\} = \mathscr{B}_2 \ominus \bar{\mathscr{B}}_2$. Let $\bar{P} = 1 - P_0$ be the projector on $\bar{\mathscr{B}}_2$. We have $P_0(\cdot) = (1/g)\mathrm{Tr}(\cdot)\mathbb{1}$. Since our transformations preserve the trace and $\mathbb{1}$, $\bar{\mathscr{B}}_2$ and $\bigvee\{\mathbb{1}\}$ are reducing subspaces for T_g, T_ℓ, $T_g T_\ell$, etc‡. Let T be such a transformation. Then

$$\bar{T} \equiv \bar{P} T \bar{P} = T\bar{P} = T - P_0 T P_0 = T - TP_0 = T - P_0 \tag{33}$$

where the last identity is a consequence of $TP_0 = P_0 = P_0 T$. The matrix representation of $T|\bar{\mathscr{A}}_g$ is therefore obtained from

$$\bar{T}_{\alpha\beta} \equiv \mathrm{Tr}(\mathbb{P}_\alpha \bar{T}(\mathbb{P}_\beta)) = \mathrm{Tr}(\mathbb{P}_\alpha T(\mathbb{P}_\beta)) - \mathrm{Tr}(\mathbb{P}_\alpha P_0(\mathbb{P}_\beta)) = T_{\alpha\beta} - 1/g \tag{34}$$

where the subtraction serves to remove the eigenvalue 1 associated with $T(\mathbb{W}_0) = \mathbb{W}_0$ (a Perron–Frobenius eigenvalue).

Now $T = T_g T_\ell$ gives $T_{\alpha\beta} = \Sigma_{n=1}^{g} \Lambda_{\alpha n}\Lambda_{\beta n}$, where $\Lambda_{\alpha n} \equiv |\langle\alpha|n\rangle|^2$ defines the matrix $(g \times g)$

$$\boldsymbol{\Lambda} = \{\Lambda_{\alpha n}\} \qquad \Lambda_{\alpha n} = |\langle\alpha|n\rangle|^2. \tag{35}$$

Thus $\boldsymbol{T} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^T$, and from (34) $\bar{\boldsymbol{T}} = \boldsymbol{T} - \boldsymbol{J}$, where $\boldsymbol{J}$ is the matrix representative of P_0. Both $\boldsymbol{T}$ and $\boldsymbol{\Lambda}$ are doubly stochastic. One can also write

$$\bar{\boldsymbol{T}} = \bar{\boldsymbol{\Lambda}}\bar{\boldsymbol{\Lambda}}^T \tag{36}$$

if one defines $\bar{\boldsymbol{\Lambda}}$ according to (34).

† Defining y_{m} so that $y_{\mathrm{m}} = x_{\mathrm{m}}$ in length.

‡ For instance: Let $\mathbb{A} \in \mathscr{B}_2$, and $\mathrm{Tr}(\mathbb{A})/g \equiv a$. Then $P_0(\mathbb{A}) = a\mathbb{1}$, $TP_0(\mathbb{A}) = T(a\mathbb{1})$, and $P_0 T(\mathbb{A}) = (1/g)\mathbb{1}\,\mathrm{Tr}(T(\mathbb{A})) = (1/g)\mathbb{1}\,\mathrm{Tr}(\mathbb{A}) = a\mathbb{1}$. All our transformations preserve the trace, and the measurements also preserve the unit operator: $T(\mathbb{1}) = \mathbb{1}$.

We do not know that $\bar{\Lambda}$ is a normal matrix. Hence the eigenvalues of $\bar{T}$, $\lambda(\bar{T})$, equal the (singular values)^2, $\mu(\bar{\Lambda})^2$, of $\bar{\Lambda}$. If $\bar{\Lambda}$ is normal, then $\lambda(\bar{T}) = |\lambda(\bar{\Lambda})|^2$. So from (31)

$$\cos^2\phi_{\rm m} = \max\lambda(\bar{T}) \qquad \text{and} \qquad \cos\phi_{\rm m} = \max|\lambda(\bar{\Lambda})| \tag{37}$$

if $\bar{\Lambda}$ is normal.

Note that the restriction to $\bar{\mathscr{A}}_g$ gets rid of the Perron–Frobenius eigenvalue of T, which implies that the minimal angle between $\mathscr{A}_g$ and $\mathscr{A}_\ell$ is always 0 because $\mathbb{1}$ is a member of both.

To find $\phi_{\rm M}$ we need $T_\ell^\perp \equiv \bar{P} - \bar{T}_\ell = 1 - T_\ell$ by (33). Then $T_g T_\ell^\perp|\bar{\mathscr{A}}_g = \bar{P}T_g(1-T_\ell)\bar{P} = \bar{T}_g - \bar{T} = T_g - T$. Since T_g is the unit operator on $\mathscr{A}_g$, from (32) $\sin^2\phi_{\rm M} = \max\lambda(\mathbf{1} - \boldsymbol{T})$, or

$$\cos^2\phi_{\rm M} = \min\lambda(\boldsymbol{T}) \qquad \text{and} \qquad \cos\phi_{\rm M} = \min|\lambda(\Lambda)| \tag{38}$$

if Λ is normal.

3.2. An example

The $g = 2$ case is worth exhibiting, although this value of g is too small to be representative. Let $\mathscr{A}_g$ be based on $\{|+\rangle, |-\rangle\}$ and $\mathscr{A}_\ell$ on $\{|\uparrow\rangle, |\downarrow\rangle\}$. $\bar{\mathscr{B}}_2$ is three dimensional, and $\bar{\mathscr{A}}_g$ and $\bar{\mathscr{A}}_\ell$ are one-dimensional subspaces. Thus $\phi_{\rm m} = \phi_{\rm M}$, which is of course not normally the case. Let $c^2 = |\langle+|\uparrow\rangle|^2$ be the maximal overlap. Both $\bar{\Lambda}$ and Λ are real/symmetric, hence normal:

$$\Lambda = \begin{pmatrix} c^2 & 1-c^2 \\ 1-c^2 & c^2 \end{pmatrix} \qquad \text{and} \qquad \bar{\Lambda} = \begin{pmatrix} c^2 - \frac{1}{2} & \frac{1}{2} - c^2 \\ \frac{1}{2} - c^2 & c^2 - \frac{1}{2} \end{pmatrix}.$$

Thus $\lambda(\Lambda) = 1,\ 2c^2 - 1$, and $\lambda(\bar{\Lambda}) = 2c^2 - 1,\ 0$. So

$$\cos\phi_{\rm m} = \cos\phi_{\rm M} = 2c^2 - 1.$$

This is also the angle in the $\mathbb{R}^3$ representation of $\bar{\mathscr{B}}_2$. Let $\mathbb{W} = |\uparrow\rangle\langle\uparrow|$ be a pure state of $\mathscr{A}_\ell$. Projecting its unit polarisation vector on the $\mathscr{A}_g$ direction we get a polarisation of length $w_+ - w_- = \langle+|\mathbb{W}|+\rangle - \langle-|\mathbb{W}|-\rangle = c^2 - (1-c^2) = \cos\phi_{\rm m}$.

3.3. Comparing aspects

As discussed in section 2.3, for perpendicular aspects $\phi_{\rm m} = \phi_{\rm M} = \pi/2$. Such aspects are—so to speak—as different as aspects can get; $\Lambda = \boldsymbol{J}$ has $g - 1$ eigenvalues equal to zero, one equal to 1. Thus $\bar{\boldsymbol{T}} = 0$.

For $\phi_{\rm M} = \pi/2$ it suffices that one $\bar{\mathbb{P}}_\alpha$ is orthogonal to one $\bar{\mathbb{P}}_n$. Since this condition may not be necessary, we might ask what is required for having $\phi_{\rm M} < \pi/2$. We shall obtain a sufficient condition. Since $\det(\boldsymbol{T}) = |\det(\Lambda)|^2$, thus, if $\det(\Lambda) \neq 0$, then $\phi_{\rm M} < \pi/2$.

Arrange the 'Greek' and 'Latin' index sets so that the maximal overlaps between $\{|\alpha\rangle\}$ and $\{|n\rangle\}$ (closest $\mathbb{P}_\alpha$ and $\mathbb{P}_n$) occur along the diagonal of Λ. Let the arrangement be

$$(\alpha, \ldots, \nu, \ldots, g)$$

$$(a, \ldots, n, \ldots, g).$$

Then $\Lambda_{\alpha a}$ is a typical diagonal element. By the Lévy-Desplangues theorem (cf Marcus and Minc 1964), $\det(\Lambda) \neq 0$ if for all α: $\Lambda_{\alpha a} > \Sigma_{n\neq a} \Lambda_{\alpha n}$ (or if $\Lambda_{\alpha a} > \Sigma_{\nu\neq\alpha} \Lambda_{\nu a}$ for all a). Since Λ is doubly stochastic we require $\Lambda_{\alpha a} > 1 - \Lambda_{\alpha a}$, or

$$\Lambda_{\alpha a} > \tfrac{1}{2} \text{ for } \textit{all } \alpha \text{ (all diagonal elements of } \Lambda) \qquad (39)$$

$$\Rightarrow \phi_{\mathrm{M}} < \pi/2.$$

For the minimal angle ϕ_{m}, standard matrix inequalities (Marcus and Minc 1964) give†

$$\max_{\alpha,\beta} |\bar{T}_{\alpha\beta}| \leq \cos^2 \phi_{\mathrm{m}} \leq \max_{\alpha} \sum_{\beta} |\bar{T}_{\alpha\beta}|. \qquad (40)$$

When $g = 2$ (section 3.2), from (39): $\phi_{\mathrm{M}} < \pi/2$ if $c^2 > \frac{1}{2}$. From (40), but applied to $\bar{\Lambda}$,

$$c^2 - \tfrac{1}{2} \leq \cos \phi_{\mathrm{m}} \leq 2c^2 - 1.$$

3.4. The exact range of (π_g, π_ℓ)

The measurement transformations T_g and T_ℓ are orthogonal projectors: $T_g^2 = T_g$, etc. On $\bar{\mathscr{B}}_2$ the purities can be expressed in the form

$$\bar{\pi}_g = \mathrm{Tr}(\bar{W}_g^2) = \mathrm{Tr}(T_g(\bar{W})^2) = \mathrm{Tr}(\bar{W} T_g(\bar{W})) = \bar{\Pi}\, \mathrm{Tr}(\bar{A} T_g(\bar{A})) \qquad (41)$$

where $\|\bar{A}\|_2 = 1$, and $\bar{A} \in \bar{\mathscr{B}}_2$-unit sphere‡. Likewise for $\bar{\pi}_\ell$. Thus $\bar{\pi}_g/\bar{\Pi}$ is in the numerical range of T_g (restricted to $\bar{\mathscr{B}}_2$), the $\bar{\mathscr{B}}_2$ expectation values of the operator T_g relative to the normalised vectors $\bar{A}$.

The joint numerical range of T_g and T_ℓ, defined as the ranges obtained with the same $\bar{A}$, was found by Lenard (1972). Hence the range of the pair $(\bar{\pi}_g, \bar{\pi}_\ell)$ is the numerical range scaled by $\bar{\Pi}$. It is the convex hull of two ellipses, given by ϕ_{m} and ϕ_{M}, and the origin $(\bar{\pi}_g, \bar{\pi}_\ell) = (0, 0)$, as illustrated in figure 1. With $\phi = \phi_{\mathrm{m}}$ or ϕ_{M} the ellipses are $(0 \leq \theta \leq 2\pi)$

$$\bar{\pi}_g = \tfrac{1}{2}\bar{\Pi}(1 + \cos(\theta + \phi)) \qquad \bar{\pi}_\ell = \tfrac{1}{2}\bar{\Pi}(1 + \cos(\theta - \phi)). \qquad (42)$$

The semi-major/minor axes are $\bar{\Pi}(1/\sqrt{2}) \cos \phi$ and $\bar{\Pi}(1/\sqrt{2}) \sin \phi$. These ellipses are confined within a square of side $\bar{\Pi}$, and centred on $(\frac{1}{2}\bar{\Pi}, \frac{1}{2}\bar{\Pi})$. Pieces of this square connect the tangent points, for instance the line from $(\bar{\Pi}, \bar{\Pi}\cos^2 \phi_{\mathrm{M}})$ to $(\bar{\Pi}, \bar{\Pi}\cos^2 \phi_{\mathrm{m}})$, as shown in figure 1.

If we choose to map the region $(\bar{\pi}_g, \bar{\pi}_\ell)$ exclusively by means of $\bar{\mathscr{B}}_2$ operators $\bar{W}$ which represent states $W \in \mathscr{S}$, then $\bar{A}$ does not cover the whole $\bar{\mathscr{B}}_2$ unit sphere unless $\bar{\Pi}$ is sufficiently small—small enough for $\bar{\mathscr{S}}$ to include a sphere of radius $\bar{\Pi}^{1/2}$. In particular, if we assert that $(\bar{\pi}_g, \bar{\pi}_\ell)$ is confined to the largest region, $\bar{\Pi} = 1 - 1/g$, corresponding to the pure states of $\bar{\mathscr{S}}$, then not all of the boundary points are accessible§.

† But using $\bar{\Lambda}$ when it is normal is sharper.

‡ For the present purposes we need consider only the subspace of $\bar{\mathscr{B}}_2$ consisting of the self-adjoint operators. The third identity in (41) is due to the duality: $\mathrm{Tr}(A^\dagger T(B)) = \mathrm{Tr}(T^*(A)^\dagger B)$, and $T_g = T_g^*$. From (22), if $\mathscr{C} = \{E_j\}$ is the cause defining T, then $\mathscr{C}_* = \{E_j^\dagger\}$ defines T^*.

§ A sphere in $\bar{\mathscr{B}}_2$ has dimension $g^2 - 2$. The boundary of the convex set $\bar{\mathscr{S}}$ also has this dimension, but is not a sphere (Bloore 1976). The pure states form a $2(g-1)$ dimensional subset of the boundary of $\bar{\mathscr{S}}$: $2(g-1)$ is the number of real coefficients in the expansion of an arbitrary $|\psi\rangle = \Sigma_\alpha c_\alpha |\alpha\rangle$, $\langle\psi|\psi\rangle = 1$. The case $g = 2$ is an exception. Also, of course, the smallest angle ϕ_{m} does not in general connect the pure states of $\bar{\mathscr{A}}_g$ and $\bar{\mathscr{A}}_\ell$.

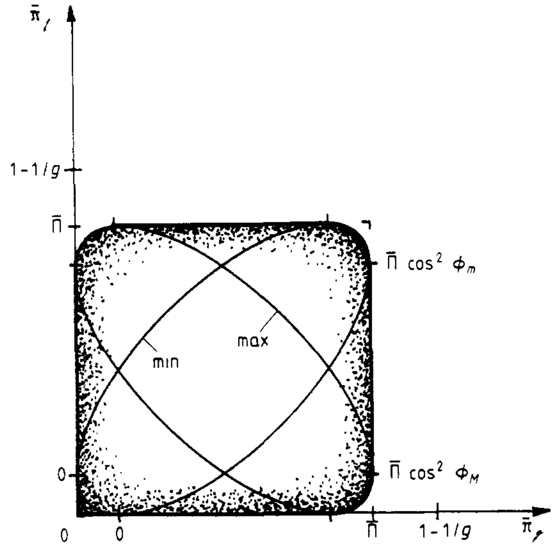

Figure 1. Accessible region for the values of the two purities $(\bar{\pi}_g, \bar{\pi}_\ell)$ associated with a pair of aspects forming the minimal angle ϕ_m and the maximal angle ϕ_M. The purity of the measured state is $\bar{\Pi} \leqslant 1-1/g$. Curved parts of the boundary are segments of ellipses given by (42). Unless the aspects have states in common $\phi_m > 0$, and the allowed region excludes the region near $(\bar{\Pi}, \bar{\Pi})$.

However, this is of little significance with respect to the relationship between the aspects $\mathscr{A}_g$ and $\mathscr{A}_\ell$. We can decide to project operators from the whole (unit) sphere of $\bar{\mathscr{B}}_2$, be they states or not. The relationship remains controlled by the two angles ϕ_m and ϕ_M.

The uncertainty relations (14) and (15) are obtained by setting $\theta = 0$, $\phi = \phi_m$ in (42). The straight line bound of (14) coincides with the exact boundary for perpendicular aspects, $\phi_m = \pi/2$:

$$\bar{\pi}_g + \bar{\pi}_\ell \leqslant \bar{\Pi} \qquad (\mathscr{A}_g \perp \mathscr{A}_\ell)$$

without the restriction to prime g of (26).

3.5. *A geometrical proof*

We use the vector notation defined in section 3.1. The vectors $\boldsymbol{W}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ span a three-dimensional subspace of $\bar{\mathscr{B}}_2$, and $\boldsymbol{x}$ and $\boldsymbol{y}$ span a two-dimensional subspace with the geometry of figure 2. Here $\boldsymbol{A}$ is the diameter through O of the circumscribed circle of the triangle $(\mathrm{O}, \boldsymbol{x}, \boldsymbol{y})$. The angle, in this plane, between $\boldsymbol{x}$ and $\boldsymbol{y}$ is given by (30).

In this three-dimensional subspace of $\bar{\mathscr{B}}_2$ $\boldsymbol{x}$ is the perpendicular projection of both $\boldsymbol{A}$ and $\boldsymbol{W}$. So $\boldsymbol{A}$ and $\boldsymbol{W}$ both point to a plane perpendicular to $\boldsymbol{x}$, through its tip. Likewise for $\boldsymbol{y}$. Thus $\boldsymbol{A}$ and $\boldsymbol{W}$ belong to the intersection of these planes, which is the line through the tips of $\boldsymbol{A}$ and $\boldsymbol{W}$, perpendicular on the $(\boldsymbol{x}, \boldsymbol{y})$ plane. Consequently, for the lengths

$$A \leqslant W. \tag{43}$$

Elementary plane geometry gives

$$|\boldsymbol{x} - \boldsymbol{y}| = A \sin\phi. \tag{44}$$

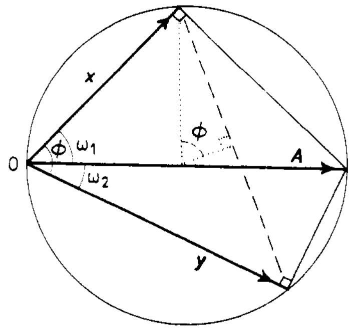

Figure 2. Geometry in the plane of $\boldsymbol{x}$ and $\boldsymbol{y}$. Dotted line construction on chord (broken line) gives (44) by standard arguments.

For later use we note that

$$xy = A^2 \cos \omega_1 \cos \omega_2 = \tfrac{1}{2}A^2(\cos \phi + \cos(\omega_1 - \omega_2)) \leq \tfrac{1}{2}A^2(\cos \phi + 1). \tag{45}$$

From (44)

$$x^2 + y^2 = A^2 \sin^2 \phi + 2xy \cos \phi$$

which for fixed A and ϕ defines an ellipse in the (x, y) plane of joint values for x and y. It is

$$x = A \cos\left(\frac{\theta + \phi}{2}\right) \qquad y = A \cos\left(\frac{\theta - \phi}{2}\right) \qquad 0 \leq \theta \leq 4\pi \tag{46}$$

or

$$\frac{\xi^2}{2\cos^2(\phi/2)} + \frac{\eta^2}{2\sin^2(\phi/2)} = A^2 \qquad \xi = \frac{1}{\sqrt{2}}(x + y) \qquad \eta = \frac{1}{\sqrt{2}}(y - x).$$

It is tangent to the bound $y = A$ at $x = A \cos \phi$, and to $x = A$ at $y = A \cos \phi$.

Consider the subset of vectors $\boldsymbol{W}$ which gives rise to the same angle ϕ. For these, (x, y) is on ellipses contained within that with $A = W$. Now allow ϕ to vary, keeping W fixed (states of given purity). The outer ellipses have the straight segments

$$y = W \qquad W \cos \phi_{\mathrm{M}} \leq x \leq W \cos \phi_{\mathrm{m}}$$

$$x = W \qquad W \cos \phi_{\mathrm{M}} \leq y \leq W \cos \phi_{\mathrm{m}}$$

as envelope. The rest of the boundary is formed by segments of the ϕ_{M} and ϕ_{m} ellipses, and the x and y axes. It is straightforward to show that this region is the same as the joint numerical range obtained by Lenard (1972).

3.6. *Fixed purity*

In order for ϕ_{m} to be certainly accessible, what is the largest purity $\bar{\Pi}$ of the initial state? A $\bar{\mathscr{B}}_2$-sphere must be inscribable in $\bar{\mathscr{S}}$. For this it is necessary and sufficient that a sphere can be inscribed in the simplex $\bar{\mathscr{S}}_g$ of an arbitrary aspect, $\mathscr{A}_g$ say. Referring to figure 3(b) in the appendix, the point closest to O on a face of $\bar{\mathscr{S}}_g$ is $(g-1)^{-1} \sum_{\alpha=1}^{g-1} \bar{\mathbb{P}}_\alpha = -(g-1)^{-1} \bar{\mathbb{P}}_g$. The purity here is $(g-1)^{-2}(1 - 1/g) = 1/g(g-1)$. Thus if $\bar{\Pi} > 1/g(g-1)$ the minimal (and maximal) angle may not be accessible.

The bound (16) is what can at present be said about the smallest angle, $\phi'_{\mathrm{m}} \geq \phi_{\mathrm{m}}$, which can occur with fixed initial $\bar{\Pi}$, pertaining to states $\bar{\mathbb{W}} \in \bar{\mathcal{S}}$.

From (45) and (30) we have

$$xy \leq \tfrac{1}{2}\bar{\Pi}\left(\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{xy}+1\right) \leq \tfrac{1}{2}\bar{\Pi}\left(\frac{c^2-1/g}{xy}+1\right). \tag{47}$$

Here the second inequality is due to $\boldsymbol{x}\cdot\boldsymbol{y} = \mathrm{Tr}(\bar{\mathbb{w}}_g \bar{\mathbb{w}}_\ell)$ being a bilinear functional over the two simplexes $\bar{\mathcal{S}}_g$ and $\bar{\mathcal{S}}_\ell$. It attains its maximum for a pair of vertices, i.e.

$$\boldsymbol{x}\cdot\boldsymbol{y} \leq \max_{\alpha,n} \mathrm{Tr}(\bar{\mathbb{P}}_\alpha \bar{\mathbb{P}}_n) = c^2 - \frac{1}{g}$$

(Marcus and Minc 1964). From (47)

$$xy \leq \tfrac{1}{4}\bar{\Pi}\left\{1+\left[1+8\frac{c^2-1/g}{\bar{\Pi}}\right]^{1/2}\right\}. \tag{48}$$

This bound is reached (if possible) at $\theta = 0$ in (46) for $A^2 = \bar{\Pi}$ if $\bar{\Pi}\cos^2(\phi/2)$ equals the upper bound in (48):

$$\cos\phi = \frac{1}{2}\left\{\left[1+8\frac{c^2-(1/g)}{\bar{\Pi}}\right]^{1/2}-1\right\}$$

As the actual $\phi'_{\mathrm{m}} \geq \phi$, this gives (16). It is a non-trivial bound if $\bar{\Pi} > c^2-(1/g)$. Expression (48) is an uncertainty relation, but not as sharp in itself as what we get using ϕ'_{m}.

3.7. A wider perspective

The difficulties in attaining the boundary for $(\bar{\pi}_g, \bar{\pi}_\ell)$ are entirely due to viewing the aspects through the subset of $\bar{\mathcal{B}}_2$ which consists of the states $\bar{\mathbb{W}}$ in $\bar{\mathcal{S}}$. There is another viewpoint. Let $T = T^* = T^2$ be any measurement transformation, such as T_g or T_ℓ. After the measurement the state becomes $\bar{\mathbb{w}} = T(\bar{\mathbb{W}})$. If $\mathbb{A}$ is an arbitrary property, $\mathbb{A} = \bar{\mathbb{A}} + (1/g)\,\mathrm{Tr}(\mathbb{A})\mathbb{1}$ shows that the significant part $\bar{\mathbb{A}}$ lies in $\bar{\mathcal{B}}_2$, whereas the second term is trivial. The mean value becomes

$$\langle \mathbb{A}\rangle = \mathrm{Tr}(\mathbb{A}\mathbb{w}) = \mathrm{Tr}(\bar{\mathbb{A}}\bar{\mathbb{w}}) + \frac{1}{g}\mathrm{Tr}(\mathbb{A})$$

and we may restrict the attention to the first term and $\bar{\mathcal{B}}_2$. Here we have a well known duality (cf (21) and (22))

$$\mathrm{Tr}(\bar{\mathbb{A}}\bar{\mathbb{w}}) = \mathrm{Tr}(\bar{\mathbb{A}}T(\bar{\mathbb{W}})) = \mathrm{Tr}(T(\bar{\mathbb{A}})T(\bar{\mathbb{W}})) = \mathrm{Tr}(T(\bar{\mathbb{A}})\bar{\mathbb{W}}).$$

Either way, the mean value is determined by the 'measured property'

$$\bar{\mathbb{a}} \equiv T(\bar{\mathbb{A}}).$$

One can therefore choose to read all of the foregoing results as pertaining to arbitrary operators in $\mathcal{B}_2$ (and $\bar{\mathcal{B}}_2$). In particular, one can read $\bar{\Pi}$ as $\|\bar{\mathbb{A}}\|_2^2$, and $\bar{\pi}_g$ as $\|\bar{\mathbb{a}}_g\|_2^2$, where $\bar{\mathbb{a}}_g \equiv T_g(\bar{\mathbb{A}})$, etc. The boundary is attainable, and in a sense we have a more direct statement about the aspects $\mathcal{A}_g$ and $\mathcal{A}_\ell$, independently of any initial state.

The allowed region for (x, y) in section 3.5 is directly the joint numerical range of the two $\mathcal{B}_2$-norms $(\|\bar{\mathbb{a}}_g\|_2, \|\bar{\mathbb{a}}_\ell\|_2)$, controlled by ϕ_{m} and ϕ_{M}. From (46) we could derive yet another uncertainty relation

$$\|\bar{\mathbb{a}}_g\|_2 + \|\bar{\mathbb{a}}_\ell\|_2 \leq 2\|\bar{\mathbb{A}}\|_2 \cos\frac{\phi_{\mathrm{m}}}{2} \qquad \bar{\mathbb{a}}_g = T_g(\bar{\mathbb{A}}) \quad \text{and} \quad \bar{\mathbb{a}}_\ell = T_\ell(\bar{\mathbb{A}}). \tag{49}$$

The reduction in magnitude of the $\bar{\mathscr{B}}_2$-norm after measurements is a measure of the 'trivialisation' of the property $\mathbb{A}$. As shown in section 2.4, if $\mathscr{A}_g \perp \mathscr{A}_\ell$ are any two perpendicular aspects, then $T_\ell(\bar{\mathbb{a}}_g) = T_g(\bar{\mathbb{a}}_\ell) = 0$. Thus the operator $T_g T_\ell = T_\ell T_g$ projects on $\bigvee\{\mathbb{1}\}$†, and trivialises every property:

$$T_g T_\ell(\mathbb{A}) = T_\ell T_g(\mathbb{A}) = \frac{1}{g}\operatorname{Tr}(\mathbb{A})\mathbb{1} \qquad (\mathscr{A}_g \perp \mathscr{A}_\ell).$$

In particular, for any $\mathbb{W} \in \mathscr{S}$

$$T_g T_\ell(\mathbb{W}) = T_\ell T_g(\mathbb{W}) = \mathbb{W}_0 \qquad (\mathscr{A}_g \perp \mathscr{A}_\ell).$$

4. Discussion

It seems appropriate to conclude with some remarks of a more general nature.

Aspects are defined with respect to the classical logic inherent in the projector set: $\mathscr{L}_g = \{\mathbb{P}_\alpha\}$ defines $\mathscr{A}_g$. Our measurement axioms were designed with the intention of defining the measurement transformation $T_g \in \mathscr{M}(\mathscr{A}_g)$ entirely with respect to this classical logic‡. In that way one secures the unambiguous communicability of the outcome, the statistic $\{w_\alpha\}$. The causal structure turns out to be defined also in classical terms—by which we mean that the set $\mathscr{C} = \{\mathbb{E}_j\}$, the objective cause of $T(\cdot) = \Sigma_j \mathbb{E}_j \cdot \mathbb{E}_j^\dagger$, is a contemporary definition of the Aristotelian 'efficient cause' in an operational context. For a measurement T_g one finds a cause $\mathscr{C}_g \subseteq \mathscr{A}_g$ expressible exclusively in terms of the operators which T_g is supposed to measure (Larsen 1988).

We have demonstrated that, with this natural extension of the standard concept of measurement a full identification of a state $\mathbb{W}$ necessarily requires the measurements of several different aspects. Sets $\mathscr{C}mp$ of complementary aspects, such as Ivanovic's perpendicular aspects, always exist and suffice. Therefore a given $\mathbb{W}$ is physically equivalent with the set of statistics which are the actual outcomes of measurements of $\mathscr{C}mp$—as well as equivalent with the set of probabilities predicted for $\mathbb{W}$ theoretically. Yet, *in no way do these conditions of operational experimentation detract from the physical reality which can be associated with the object state* $\mathbb{W}$. Each $\mathbb{W}$ belongs to at least one aspect. A pure state, $\mathbb{P}_\psi$ say, belongs to all the different aspects whose $\mathscr{L}$ contain $\mathbb{P}_\psi$ (i.e. aspects whose $\bar{\mathscr{B}}_2$ parts intersect along $\bar{\mathbb{P}}_\psi$): one can choose an arbitrary basis for the part of $\mathbb{H}$ which is orthogonal to $|\psi\rangle$, and these choices correspond to different aspects.

The aspects of $\mathbb{W}$ may not be known. Even if known, an experimenter may find it beyond his means to perform a reproducing measurement of any of them. But this is not a matter of principle. Rather, it is a situation one must expect to encounter—especially in dealing with unique, spontaneously arising states. In principle, any of the aspects of $\mathbb{W}$ could be measured—themselves providing the requisite operational cause $\mathscr{C}$. Doing so would allow the experimenter to confirm that $\mathbb{W}$ did indeed belong to the aspect investigated, without altering the state $\mathbb{W}$. We therefore assert that, a complete respresentation of the objective reality associated with $\mathbb{W}$ is to be found in and among the aspect(s) to which $\mathbb{W}$ belongs.

If one considers more general transformations than measurements, to draw causes from the aspects of a state $\mathbb{W}$ will produce transformations which leave $\mathbb{W}$ invariant.

† That is, $T_g T_\ell = T_\ell T_g = P_0 = (1/g)\operatorname{Tr}(\cdot)\mathbb{1}$.
‡ The redundancy is provisional.

This is related to the contemporary viewpoint that physical reality is linked to symmetry. More specifically, when invariances imply conservation laws physical reality can be associated with definite quantitative values of certain sets of properties. For instance, the index $\psi \in \{\alpha\}$ of a definite pure state $\mathbb{P}_\psi \in \mathscr{A}_\varphi$ is synonymous with a definite set of values of a so-called complete set of mutually commuting operators belonging to $\mathscr{A}_\varphi$.

Let us just remark upon one side of this standard interpretation. It might be asked how one distinguishes operationally between the different members of $\{\alpha\}$, since as far as $\mathscr{A}_\varphi$ is concerned all pure states $\mathbb{P}_\alpha \in \mathscr{L}_\varphi$ have the same invariances. However, each $\mathbb{P}_\alpha$ belongs to other aspects than $\mathscr{A}_\varphi$—and these different aspects are not all the same for any pair of $\mathbb{P}_\alpha \neq \mathbb{P}_\beta$. Thus one can, in principle at least, distinguish α from β by certain invariances of $\mathbb{P}_\alpha$ and $\mathbb{P}_\beta$ which do not coincide. The causes defining these 'secondary' invariance transformations come from aspects other than $\mathscr{A}_\varphi$. However exotic they may appear, in principle they suffice for the operational distinction between the members of $\mathscr{L}_\varphi$. Referring to figure 3 we would say that this way of distinguishing, between $\bar{\mathbb{P}}_\alpha$ and $\bar{\mathbb{P}}_\beta$ say, is geometrically self-evident†.

Elsewhere we have discussed the probabilistic nature of the theoretical predictions made with a given $\mathbb{W}$ (Larsen 1988). This probabilistic/statistical theme emerges whenever it is considered to measure aspects to which $\mathbb{W}$ does not belong. Its precise characterisation was the subject of the present investigation. We argued, in connection with the objective causality based on the existence of an operational cause $\mathscr{C}$ for any transformation, that the statistical dispersion in the outcome $\{w_\alpha\}$, say, can be ascribed to the alterations of $\mathbb{W}$ which ensue from measuring $\mathscr{A}_\varphi$, if $\mathscr{A}_\varphi$ is foreign to $\mathbb{W}$. These causal influences from a future measurement environment are foreseen in the predicted probabilities $\{P(\alpha)\}$, while the state remains $\mathbb{W}$. That the consequences of such hypothetical future events include statistical dispersion in the data one obtains can be seen as the price one pays for acquiring an image of $\mathbb{W}$, projected on a simple and intelligible aspect $\mathscr{A}_\varphi \in \mathscr{C}mp$.

We do not find that these experimental opportunities to measure sets $\mathscr{C}mp$ of complementary aspects of our choice need reflect on the concrete reality of the state $\mathbb{W}$ at hand initially. It is immaterial that the measurements of the members of $\mathscr{C}mp$ must needs be mutually exclusive, operationally. And it is immaterial that there may be statistical dispersion in the data, because there are external influences on the way from $\mathbb{W}$ to $\mathbb{w}_\varphi$, as described objectively by the cause $\mathscr{C}_\varphi$ defining $T_\varphi \in \mathscr{M}(\mathscr{A}_\varphi)$. Perhaps one may see these remarks as a way to reconcile objective complementary and objective reality in a contemporary operational context.

Appendix. Superspace geometry

The set of states $\mathscr{S}$ is a convex subset of the Hilbert space $\mathscr{B}_2$ (=Hilbert-Schmidt operators on $\mathbb{H}$) with inner product $\mathrm{Tr}(\mathbb{A}^+\mathbb{B})$. Every element of $\mathscr{B}_2$, hence every state $\mathbb{W} \in \mathscr{S}$, can be expressed as a (countable) linear combination of operators belonging to a basis for $\mathscr{B}_2$. An orthonormal basis for $\mathscr{B}_2$ may consist of self-adjoint operators which are compact and normal, hence diagonalisable (Conway 1985). Therefore each element of a self-adjoint basis belongs to an aspect, but not all to the same aspect.

† Of course, in practical work it is much more convenient to investigate how a given state is changed in transformations which are not invariances. But this is less relevant to the question of physical reality in the strict sense.

The intersection, $\mathscr{A} \cap \mathscr{B}_2$, of an aspect $\mathscr{A}$ with $\mathscr{B}_2$ is subspace of $\mathscr{B}_2$, with its defining projector set $\mathscr{L}$ as one orthonormal, self-adjoint basis. If $\dim(\mathbb{H}) \equiv g < \infty$, then $\dim(\mathscr{B}_2) = g^2$ and $\dim(\mathscr{A} \cap \mathscr{B}_2) = g$. For simplicity we refer to $\mathscr{A} \cap \mathscr{B}_2$ as 'the aspect $\mathscr{A}$' in the geometrical context, although not all of $\mathscr{A}$ is in $\mathscr{B}_2$ if $g = \infty$ (in particular, $\mathbb{1}$ is not in $\mathscr{B}_2$ when $g = \infty$, although $\mathbb{1} \in \mathscr{A}$).

The linear subspace $\bigvee\{\mathbb{1}\}$ is in $\mathscr{B}_2$ when $g < \infty$ (as we assume henceforth), and is orthogonal to the stack of (g^2-1)-dimensional hyperplanes defined by $\mathrm{Tr}(\mathbb{A}) =$ constant. Let

$$\bar{\mathscr{B}}_2 \equiv \left\{ \bar{\mathbb{A}} \equiv \mathbb{A} - \frac{1}{g}\mathrm{Tr}(\mathbb{A})\mathbb{1} \right\} \qquad \mathscr{B}_2 = \bigvee\{\mathbb{1}, \bar{\mathscr{B}}_2\} \tag{A1}$$

be the traceless operator hyperplane. Each aspect intersects $\bar{\mathscr{B}}_2$ in a $(g-1)$-dimensional subspace (because $\bigvee\{\mathbb{1}\}$ belongs to every aspect: $\mathbb{1} = \Sigma_\alpha \mathbb{P}_\alpha \in \mathscr{A}_g$, etc). If $\mathbb{A} \in \mathscr{A}$, then $\bar{\mathbb{A}} \in \bar{\mathscr{A}} \equiv \mathscr{A} \cap \bar{\mathscr{B}}_2$.

Consider a definite aspect $\mathscr{A}_g$. Then $\mathscr{S}$ intersects $\mathscr{A}_g$ in a convex polyhedron with the pure states $\mathscr{L}_g = \{\mathbb{P}_\alpha\}$ at the vertices (i.e. $\mathbb{W}_g = \Sigma_\alpha w_\alpha \mathbb{P}_\alpha$, $w_\alpha \geqslant 0$, $\Sigma_\alpha w_\alpha = 1$ for $\mathbb{W}_g \in \mathscr{A}_g$). In $\bar{\mathscr{B}}_2$ the intersection of $\bar{\mathscr{A}}_g$ with $\mathscr{S}$, called $\bar{\mathscr{S}}_g \equiv \bar{\mathscr{A}}_g \cap \mathscr{S}$, is the corresponding $(g-1)$-dimensional simplex with vertices $\{\bar{\mathbb{P}}_\alpha\}$, of which $g-1$ are linearly independent. With lengths given by the $\mathscr{B}_2$-norm $\|\cdot\|_2$, these configurations are indicated in figure 3.

Let $\{\bar{\mathbb{A}}^{(k)} | k = 1, \ldots, g^2-1\}$ be a self-adjoint orhonormal basis for $\bar{\mathscr{B}}_2$, known as a *quorum*† (Band and Park 1970, 1971, Park and Band 1971). Thus $\mathbb{W} = (1/g)\mathbb{1} + \Sigma_{k=1}^{g^2-1} \langle \bar{\mathbb{A}}^{(k)} \rangle \bar{\mathbb{A}}^{(k)}$, where $\langle \bar{\mathbb{A}}^{(k)} \rangle = \mathrm{Tr}(\mathbb{W}\bar{\mathbb{A}}^{(k)})$‡.

Other examples of basis sets for $\mathscr{B}_2$ when $g < \infty$ are given in Fano (1957) and Schwinger (1960a, b). For $g = \infty$ the geometry is related to coherent states (Klauder and Sudarshan 1968, Cahill and Glauber 1969), and the Wigner/Weyl representation (Schwinger 1960b, Band and Park 1979, Royer 1985, Fano 1957). Analogous representations for $g < \infty$ are given by Stratonovich (1957), Radcliffe (1971) and Wootters (1986b, 1987).

In order to set up mathematical representations we need only one aspect, say $\mathscr{A}_g$. The wavefunction $\langle \alpha | \psi \rangle = \psi(\alpha)$ represents the pure state $|\psi\rangle$, or $\mathbb{P}_\psi = |\psi\rangle\langle\psi|$. But if one measures $\mathscr{A}_g$ one acquires merely the information held in $w_\alpha = P(\alpha) = |\psi(\alpha)|^2$. Likewise the density matrix $W_{\alpha\alpha'} = \langle \alpha | \mathbb{W} | \alpha' \rangle$ represents the mixed state $\mathbb{W}$, whereas a measurement of $\mathscr{A}_g$ yields merely $w_\alpha = P(\alpha) = W_{\alpha\alpha}$.

In the context where the Wigner function occurs one uses two aspects in order to create a representation of $\mathscr{B}_2$, say $\mathscr{A}_g$ and $\mathscr{A}_\ell$. Suppose that for all α, $n : \langle \alpha | n \rangle \neq 0$. Then $\{\mathbb{P}_\alpha \mathbb{P}_n\}$ is an orthogonal basis in $\mathscr{B}_2$ (not a self-adjoint one, and not normalised). So $W(\alpha, n) = \mathrm{Tr}(\mathbb{W}\mathbb{P}_\alpha\mathbb{P}_n)$ represents $\mathbb{W}$ in terms of a function of two variables, α and n, which suffices because $g \times g = g^2 = \dim(\mathscr{B}_2)$, essentially. With another choice of basis the Wigner function can likewise be a function of two real (phasespace) variables pertaining to two different aspects (Wootters 1986b, 1987).

But $W(\alpha, n)$ is not the outcome of any measurement, just as $\psi(\alpha)$ is not, and neither is positive definite (or real valued). To detect all the information held in either $|\psi\rangle$ or $\mathbb{W}$, as well as in any representation like $\psi(\alpha)$ or $W(\alpha, n)$, one must measure many more different aspects than those needed to create the representation.

† More generally, to have a quorum we need merely a linearly independent set from $\bar{\mathscr{B}}_2$.

‡ In $g = 2$: $\bar{\mathscr{B}}_2$ is three dimensional; $\{\bar{\mathbb{A}}^{(k)}\} = \{(1/\sqrt{2})\sigma_x, (1/\sqrt{2})\sigma_y, (1/\sqrt{2})\sigma_z\}$, $\mathbb{W} = \frac{1}{2}(\mathbb{1} + \vec{P}\cdot\vec{\sigma})$, $\mathscr{A}_x = \bigvee\{(1/\sqrt{2})\mathbb{1}, (1/\sqrt{2})\sigma_x\}$, and $\bar{\mathscr{A}}_x = \bigvee\{(1/\sqrt{2})\sigma_x\}$.

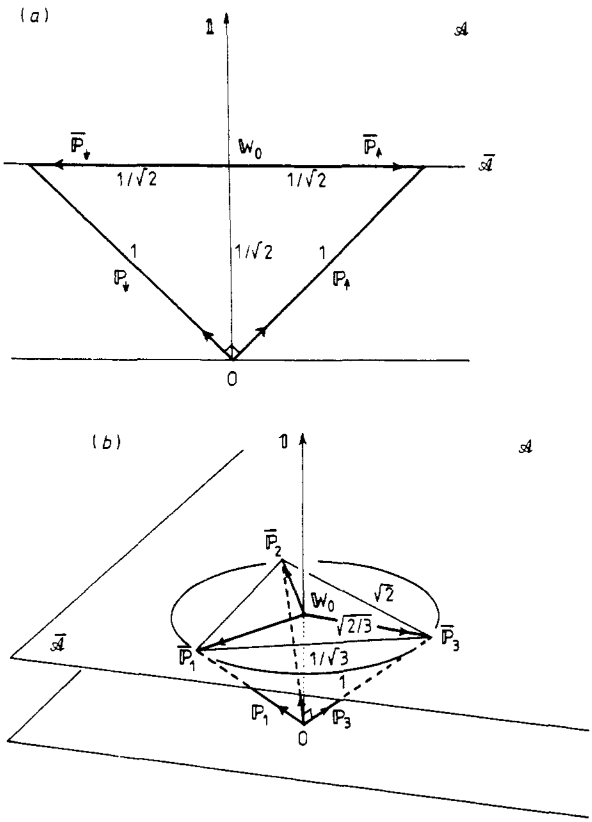

Figure 3. Subspaces of $\mathscr{B}_2$ and $\bar{\mathscr{B}}_2$ associated with a given aspect (*a*) for $g = 2$, and (*b*) for $g = 3$. All states belonging to the aspect $\mathscr{A}$ are in the polyhedra with the projectors of $\mathscr{A}$ at the vertices, while $\bar{\mathscr{A}} = \mathscr{A} \cap \bar{\mathscr{B}}_2$ contains the simplexes corresponding to the face which contains $\mathbb{W}_0$. All lengths (and angles) are defined by the norm $\|\cdot\|_2$ given by the inner product $\mathrm{Tr}(\cdot^\dagger \cdot)$. The geometry is Euclidean. Note that, while the $\mathbb{P}$ are orthogonal, the $\bar{\mathbb{P}}$ are not. There is g-fold symmetry about the $\mathbb{1}$-axis.

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